General Certificate of Education
June 2009
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Wednesday 17 June 2009 9.00 am to 10.30 am

For this paper you must have: • a 12-page answer book • the blue AQA booklet of formulae and statistical tables. You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Let $\mathbf{P} = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} k & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$, where k is a constant.

(a) Determine the product matrix $\mathbf{PQ}$, giving its elements in terms of k where appropriate. *(3 marks)*

(b) Find the value of k for which $\mathbf{PQ}$ is singular. *(2 marks)*

2 (a) Write down the 3×3 matrices which represent the transformations A and B, where:

(i) A is a reflection in the plane $y = x$; *(2 marks)*

(ii) B is a rotation about the z-axis through the angle θ, where $\theta = \dfrac{\pi}{2}$. *(1 mark)*

(b) (i) Find the matrix $\mathbf{R}$ which represents the composite transformation

'A followed by B' *(3 marks)*

(ii) Describe the single transformation represented by $\mathbf{R}$. *(2 marks)*

3 The plane Π has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(b) Show that the line with equation $\mathbf{r} = \begin{bmatrix} 7 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix}$ does not intersect Π, and explain the geometrical significance of this result. *(4 marks)*

4 (a) Show that the system of equations

$$\begin{aligned} 3x - y + 3z &= 11 \\ 4x + y - 5z &= 17 \\ 5x - 4y + 14z &= 16 \end{aligned}$$

does not have a unique solution and is consistent.

(You are not required to find any solutions to this system of equations.) *(4 marks)*

(b) A transformation T of three-dimensional space maps points (x, y, z) onto image points (x', y', z') such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x - y + 3z - 2 \\ 2x + 6y - 4z + 12 \\ 4x + 11y + 4z - 30 \end{bmatrix}$$

Find the coordinates of the invariant point of T. *(8 marks)*

5 The points A, B, C and D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively, relative to the origin O, where

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} 5 \\ 5 \\ 11 \end{bmatrix}$$

(a) Using scalar triple products:

(i) show that $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are coplanar; *(2 marks)*

(ii) find the volume of the parallelepiped defined by AB, AC and AD. *(4 marks)*

(b) (i) Find the direction ratios of the line BD. *(2 marks)*

(ii) Deduce the direction cosines of the line BD. *(2 marks)*

6 The plane transformation T is defined by

$$\mathrm{T} : \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\mathbf{M} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$.

(a) Evaluate $\det \mathbf{M}$ and state the significance of this answer in relation to T. *(2 marks)*

(b) Find the single eigenvalue of **M** and a corresponding eigenvector. Describe the geometrical significance of these answers in relation to T. *(5 marks)*

(c) Show that the image of the line $y = \frac{1}{2}x + k$ under T is $y' = \frac{1}{2}x' + k$. *(3 marks)*

(d) Given that T is a shear, give a full geometrical description of this transformation. *(2 marks)*

7 The 2×2 matrix **M** has an eigenvalue 3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and a second eigenvalue -3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

The diagonalised form of **M** is $\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$.

(a) (i) Write down suitable matrices **D** and **U**, and find $\mathbf{U}^{-1}$. *(4 marks)*

(ii) Hence determine the matrix **M**. *(3 marks)*

(b) Given that n is a positive integer, use the result $\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$ to show that:

(i) when n is even, $\mathbf{M}^n = 3^n\,\mathbf{I}$;

(ii) when n is odd, $\mathbf{M}^n = 3^{n-1}\,\mathbf{M}$. *(6 marks)*

8 (a) Matrix $\mathbf{M} = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$. Without attempting to factorise, expand fully $\det \mathbf{M}$.

(2 marks)

(b) Matrix $\mathbf{N} = \begin{bmatrix} d & e & f \\ f & d & e \\ e & f & d \end{bmatrix}$. Find the product matrix $\mathbf{MN}$. *(3 marks)*

(c) Prove that the product

$$(a^3 + b^3 + c^3 - 3abc)(d^3 + e^3 + f^3 - 3def)$$

can be written in the form $x^3 + y^3 + z^3 - 3xyz$, stating clearly each of x, y and z in terms of a, b, c, d, e and f. *(2 marks)*

END OF QUESTIONS

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